A Column Generation approach for a Multi-Activity Tour Scheduling Problem

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Outline

Context

Human Capital Management:

- talent acquisition and management (recruitment, development)
- workforce management (planning optimization, time and attendance)

What is a multi-activity shift scheduling problem?

Multi-activity shift scheduling problems consist in assigning employees to activities over a given time horizon, taking into account different constraints:

- organizational constraints
- legal constraints
- social constraints ...

and optimizing a given objective:

- minimize costs
- maximize equity

Application areas

Some applications areas:

- restaurants
- healthcare systems
- retail stores
- call centres
- hotels
- transportation systems
- ...

Some references

Alfares, H. K. (2004). Survey, Categorization, and Comparison of Recent Tour Scheduling Literature. *Annals of Operations Research*, 127(1-4):145–175.

Ernst, A. T., Jiang, H., Krishnamoorthy, M., Owens, B., and Sier, D. (2004a). An Annotated Bibliography of Personnel Scheduling and Rostering. *Annals of Operations Research*, 127(1-4):21–144.

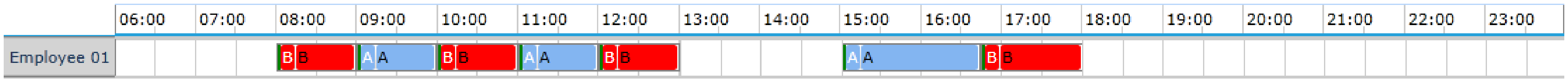
Ernst, A. T., Jiang, H., Krishnamoorthy, M., and Sier, D. (2004b). Staff scheduling and rostering: A review of applications, methods and models. *European Journal of Operational Research*, 153(1):3–27.

Van den Bergh, J., Beliën, J., De Bruecker, P., Demeulemeester, E., and De Boeck, L. (2013). Personnel scheduling: A literature review. *European Journal of Operational Research*, 226(3):367–385.

...

What is a schedule?

- **Time horizon**: is given by a fixed number of consecutive days (ex: 1 day, 1 week)
- **Slots**: each day is divided into the same number of successive time periods of equal length (ex: 15 min, 30 min, 1 hour)
- **Schedule**: sequence of assigned slots that cover the time horizon for a given employee



The considered problem

→ Workload constraints

→ Skills constraints

→ Legal constraints:

- daily working hours
- consecutive working hours
- amplitude of working day
- activities' duration
- breaks type and duration
- rest period between working day
- weekly working hours
- consecutive working days

Dantzig-Wolfe formulation

Variables:

- $x_j = 1$ if schedule j is selected, 0 otherwise
- z_k and v_k are the under and over assignments

$$(P) \quad \min \sum_{j \in N} c_j x_j + \sum_{k \in K} (\underline{c}_k z_k + \bar{c}_k v_k)$$

$$\text{s.t.} \sum_{j \in N_k} x_j + z_k - v_k = b_k \qquad \forall k \in K \qquad \text{(wkl)}$$

$$\sum_{j \in N_i} x_j = 1 \qquad \forall i \in I \qquad \text{(conv)}$$

$$x_j \in \{0, 1\} \qquad \forall j \in N$$

$$z_k, v_k \geq 0 \qquad \forall k \in K$$

Dantzig-Wolfe formulation

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$$
\begin{aligned}
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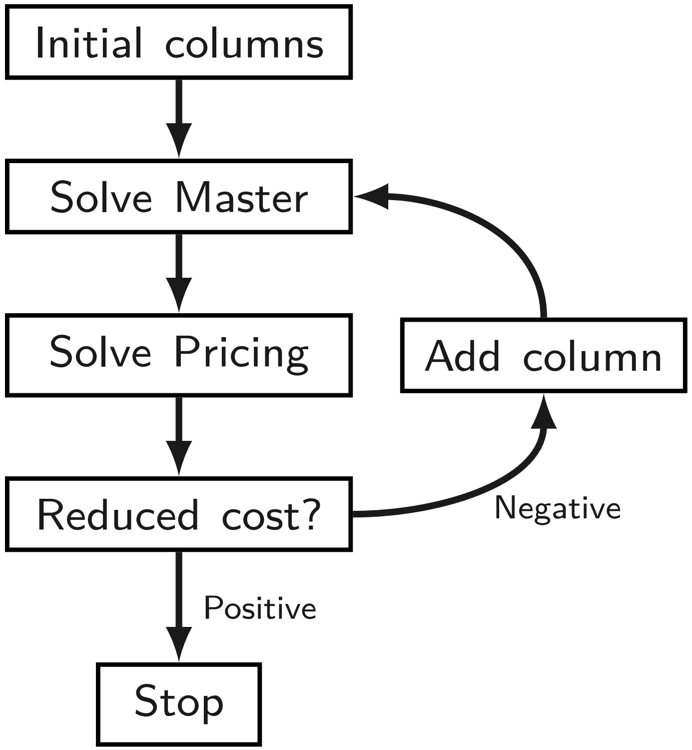
& \sum_{j \in N_i} x_j = 1 & \forall i \in I \quad & \text{(conv)} \\
& x_j \in \{0, 1\} & \forall j \in N & \\
& z_k, v_k \geq 0 & \forall k \in K &
\end{aligned}
$$

$$
x_j = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} K \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} I \end{matrix}
$$

Column generation

→ Avoid the complete enumeration of the variables.

- Solve the master problem
  → get dual variables
- Solve the pricing problem
  → generate new columns
- Stop when no columns with negative reduced costs are generated.



Dual ascent heuristic

→ **Goal**: estimation of the optimal dual variables

→ **Steps**:[1]

(1) parametric relaxation of the master problem (P)

(2) Lagrangian relaxation of the equality constraints (wkl)-(conv)

(3) subgradient method to solve the Lagrangian dual problem

[1] Boschetti, M. A., Mingozzi, A., and Ricciardelli, S. (2008). A dual ascent procedure for the set partitioning problem. *Discrete Optimization*, 5(4):735-747.

Dual ascent heuristic: comparison

Boschetti et al.

$$\min \sum_{j \in N} c_j x_j$$

$$\sum_{j \in N_i} x_j = 1 \qquad \forall i$$

$$x_j \in \{0,1\} \qquad \forall j$$

- Parametric + Lagrangian relaxation
- Stronger than classical Lagrangian relaxation
- Dual variables $u_i = \frac{q_i(c_{j_i} - \lambda_i(R_{j_i}))}{q(R_{j_i})} + \lambda_i$

Our problem

$$\min \sum_{j \in N} c_j x_j + \sum_{k \in K} (\underline{c}_k z_k + \bar{c}_k v_k)$$

$$\sum_{j \in N_k} x_j + z_k - v_k = b_k \qquad \forall k$$

$$\sum_{j \in N_i} x_j = 1 \qquad \forall i$$

$$x_j \in \{0,1\} \qquad \forall j$$

$$z_k, v_k \geq 0 \qquad \forall k$$

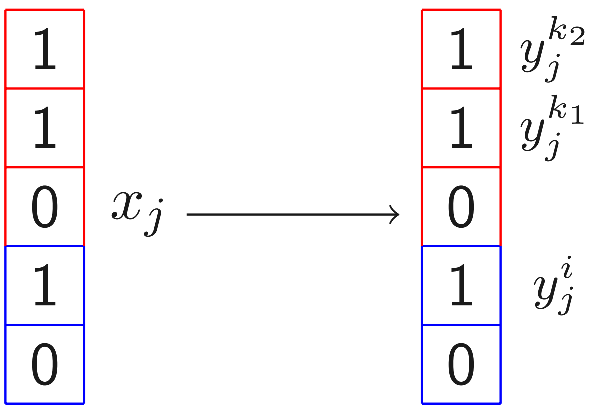
- Right hand side b_k different of 1
- Negative coefficients
- Continuous variables z_k, v_k

(1) Parametric relaxation

Each variables $x_j \in \{0, 1\}$ is substituted according the expression

$$x_j = \frac{1}{|R_j|}\left(\sum_{k \in R_j \cap K} y_j^k + y_j^i\right)$$

where $y_j^k, y_j^i \in \{0, 1\}$ and $R_j = \{\text{indices of rows covered by } j\}$



(2) Lagrangian relaxation

We relax constraints (wkl) and (conv) by means of lagrangian multipliers $\lambda \in \mathbb{R}^{|K|}$ and $\mu \in \mathbb{R}^{|I|}$.

$$(R) \quad \min \sum_{j \in N} c_j \frac{1}{|R_j|} \left(\sum_{k \in R_j} y_j^k + y_j^i \right) + \sum_{k \in K} (\underline{c}_k z_k + \bar{c}_k v_k)$$

$$\text{s.t.} \sum_{j \in N_k} \frac{1}{|R_j|} \left(\sum_{k' \in R_j} y_j^{k'} + y_j^i \right) + z_k - v_k = b_k \qquad \forall k \qquad \text{(wkl)}$$

$$\sum_{j \in N_i} \frac{1}{|R_j|} \left(\sum_{k \in R_j} y_j^k + y_j^i \right) = 1 \qquad \forall i \qquad \text{(conv)}$$

$$y_j^k, y_j^i \in \{0, 1\} \qquad \forall j, k, i$$

$$z_k, v_k \geq 0 \qquad \forall k$$

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$$y_j^k, y_j^i \in \{0, 1\} \qquad \forall j, k, i$$

$$z_k, v_k \geq 0 \qquad \forall k$$

(2) Lagrangian relaxation

We get the Lagrangian subproblem $LR(\lambda, \mu)$

$$z_{LR}(\lambda,\mu) = \min \sum_{k\in K} \left(\sum_{j\in N_k} c_j^{\lambda\mu} y_j^k + \underline{c}_k^{\lambda} z_k + \bar{c}_k^{\lambda} v_k + b_k \lambda_k \right) + \sum_{i\in I} \left(\sum_{j\in N_i} c_j^{\lambda\mu} y_j^i + \mu_i \right)$$

(wkl)

(conv)

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$LR(\lambda, \mu)$ is **decomposable** in $|K| + |I|$ subproblems.

Lagrangian suproblem k:

$$z_{LR_k}(\lambda,\mu) = \min \sum_{j\in N_k} c_j^{\lambda\mu} y_j^k + \underline{c}_k^{\lambda} z_k + \bar{c}_k^{\lambda} v_k$$

$$\sum_{j\in N_k} y_j^k + z_k - v_k = b_k$$

$$y_j^k \in \{0,1\}, z_k, v_k \geq 0$$

Solution (ordering $c_j^{\lambda\mu}$)

→ Ex: $-\bar{c}_k^{\lambda} \leq c_{j_1}^{\lambda\mu} \leq \cdots \leq c_{j_{b_k}}^{\lambda\mu} \leq \underline{c}_k^{\lambda}$

$$\begin{cases} y_j^k = 1 & j = j_1,\ldots,j_{b_k} \\ z_k = 0 \\ v_k = 0 \end{cases}$$

Dual of Lagrangian subproblem k:

$$z_{DLR_k}(\lambda,\mu) = \max b_k \tilde{u}_k - \sum_{j\in N_k} \tilde{w}_j^k$$

$$\tilde{u}_k - \tilde{w}_j^k \leq c_j^{\lambda\mu}$$

$$-\bar{c}_k^{\lambda} \leq \tilde{u}_k \leq \underline{c}_k^{\lambda}$$

$$\tilde{w}_j^k \geq 0.$$

Solution

$$\begin{cases} \tilde{u}_k = c_{j_{b_k}}^{\lambda\mu} \\ \tilde{w}_j^k = c_{j_{b_k}}^{\lambda\mu} - c_j^{\lambda\mu}, & j = j_1,\ldots,j_{b_k} \\ \tilde{w}_j^k = 0 \end{cases}$$

(2) Lagrangian relaxation: solving the Lagrangian subproblem

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(2) Lagrangian relaxation: get dual feasible solution

→ We have a solution of the Lagrangian subproblem $LR(\lambda, \mu)$

→ Using $(\tilde{u}, \tilde{t}, \tilde{w})$ we can get a dual feasible solution of the original problem

$$(P) \ \min \sum_{j \in N} c_j x_j + \sum_{k \in K} (\underline{c}_k z_k + \bar{c}_k v_k)$$

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$$(D) \ \max z_D = \sum_{k \in K} b_k u_k + \sum_{i \in I} t_i$$

$$\sum_{k \in R_j} u_k + t_i \leq c_j \qquad \forall j$$

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Dual feasible solution (u, t): $\begin{cases} u_k = \tilde{u}_k + \lambda_k & \forall k \\ t_i = \tilde{t}_i - \max_{j \in N_i} \{\sum_{k \in R_j} \tilde{w}_j^k\} + \mu_i & \forall i \end{cases}$

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(2) Lagrangian relaxation: some properties

→ The solution (u, t) is dual feasible and has a value z_D

$$z_{LR}(\lambda, \mu) \leq z_D \leq z_D^*$$

→ The bound $z_{LR}(\lambda, \mu)$ is stronger than the one given by the classical Lagrangian relaxation

$$z_{CLR}(\lambda, \mu) \leq z_{LR}(\lambda, \mu)$$

→ Due to the integrality property and $z_{CLR}(\lambda, \mu) \leq z_{LR}(\lambda, \mu) \leq z_D^*$

$$\max_{\lambda,\mu} z_{CLR}(\lambda, \mu) = \max_{\lambda,\mu} z_{LR}(\lambda, \mu) = z_D^*$$

The problem $\max_{\lambda,\mu} z_{LR}(\lambda, \mu)$ is called **Lagrangian dual**

→ We need to solve the Lagrangian dual

$$\max_{\lambda,\mu} z_{LR}(\lambda,\mu)$$

→ We use the **subgradient method**:

(S1) Set $k = 0$ and choose initials (λ, μ)
(S2) Solve $LR(\lambda, \mu)$ and obtain the solution (y, z, v)
(S3) If *stopping criteria* is satisfied, stop
(S4) Compute $\lambda = \lambda + \alpha_\lambda g_\lambda$ and $\mu = \mu + \alpha_\mu g_\mu$
where g_λ and g_μ are the subgradients
(S5) Set $k = k + 1$ and go to (S2)

Computational results

Data sets:

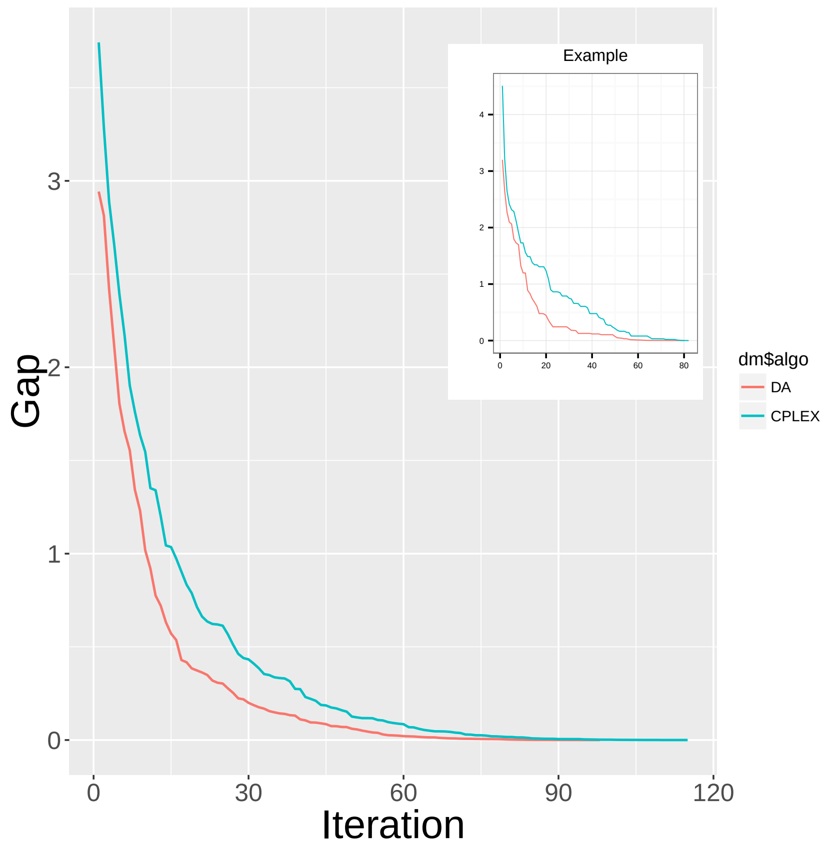
(1) OR-Library set-covering and set-partitioning instances (adapted)

(2) Personnel scheduling instances provided by Horizontal Software

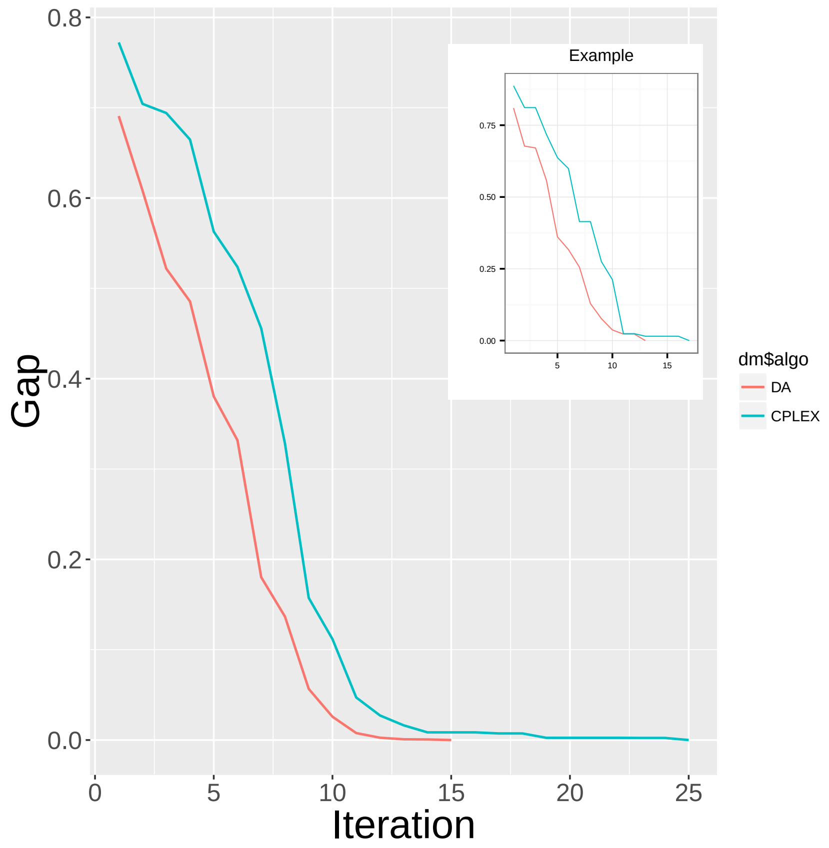
- Time horizon of 1 day
- Slots 30 minutes or 1 hour
- 33 or 66 employees
- 10 activities

Gap: $\frac{z^*_{LP} - z_{ALGO}}{z^*_{LP}}$

OR-Library



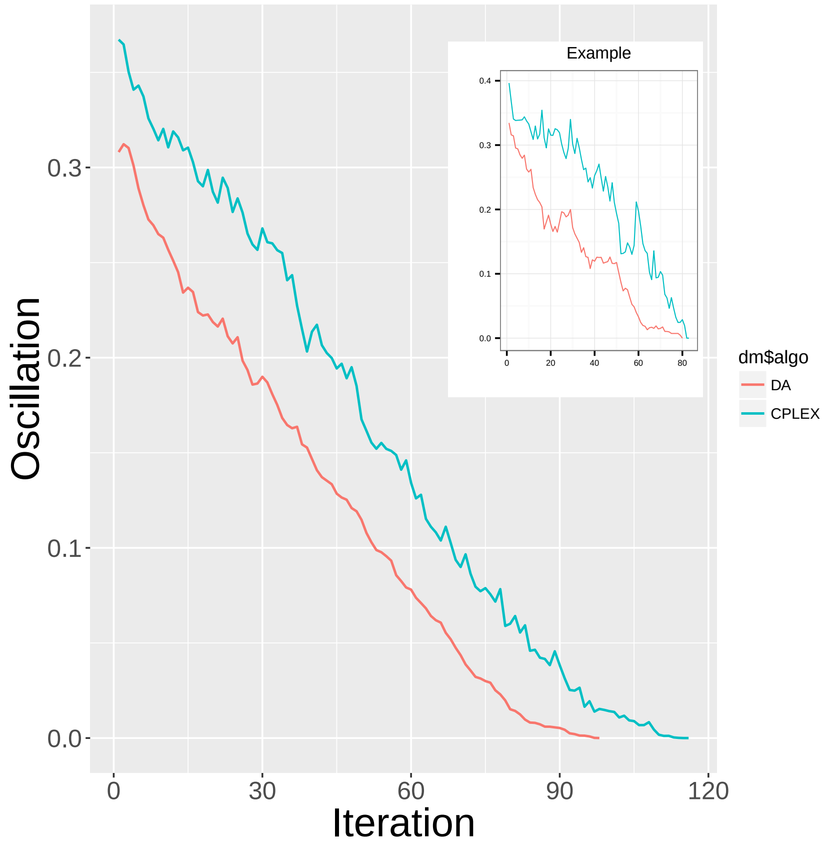
Personnel Scheduling



Computational results

Dual variables oscillations: $\frac{||(u^*,t^*)-(u,t)_{ALGO}||_2}{||(u^*,t^*)||_2}$

OR-Library



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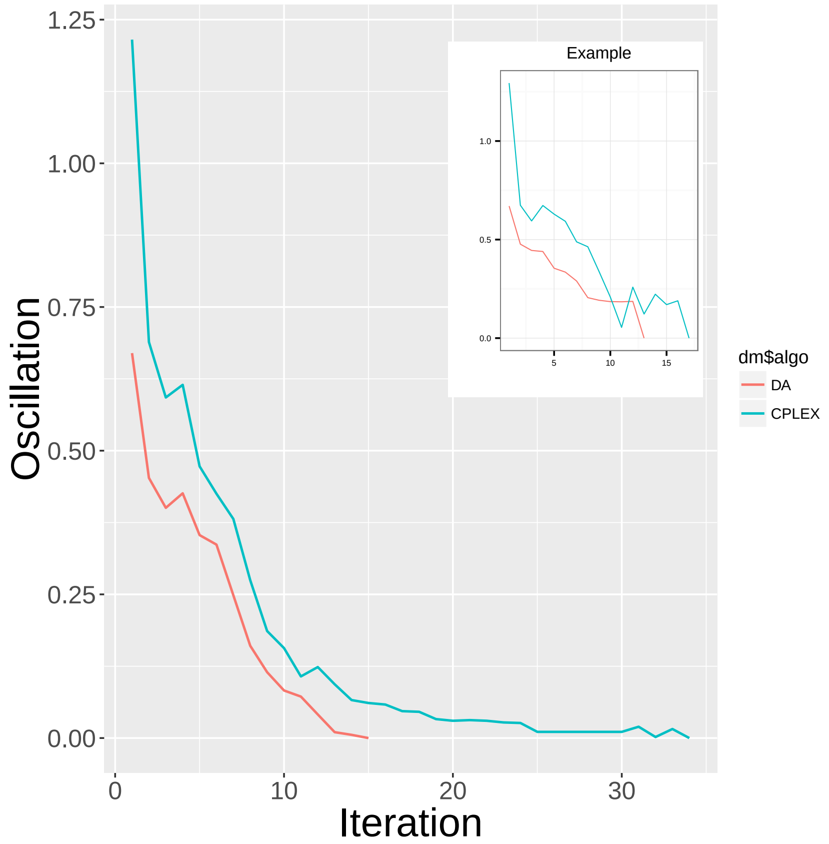

Conclusions and perspectives

Conclusions

- We propose a dual ascent heuristic to estimate the optimal dual variables
- The tailing-off effect is softened and the dual variables are more stable compared to Cplex

Perspectives

- Develop an algorithm to solve efficiently the pricing problem for large instances
- Develop a complete method to find an integer solution (branch-and-price)

Thank you for your attention!